MA 126 — Fall 2016 — Prof. Clontz — Midterm

Name: Solutions

- **Choose SEVEN of the eight non-bonus problems to solve. Mark the one you don't want graded by marking it with the word "SKIP" in the upperleft corner of the page.** If you fail to do so, you will be penalized 15 points.
- Show all of your work and give a complete solution. When in doubt, don't skip any steps. Partial credit will be given at the discretion of the professor.
- No partial credit will be given for bonus questions.
- You may use at most three pages (front and back) of 8.5×11 inch paper of notes in your own handwriting.
- No personal electronics are allowed. You may borrow a calculator from the instructor, but none of the questions require the use of a calculator.
- This exam is due after 50 minutes. Tests submitted over one minute late will not be graded.
- Your total score for this midterm will be multiplied by 3 and used as your Midterm grade for the semester.

1. (15 points) Use this definition:

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

to prove the following identity:

$$(\cosh x)^2 = \frac{1}{2}\cosh 2x + \frac{1}{2}.$$

$$(\cosh x)^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2\cancel{e^x e^{-x}} + e^{-2x}}{4}$$

$$= \frac{e^{2x} + e^{-2x}}{4} + \frac{1}{2}$$

$$\frac{1}{2}\cosh 2x + \frac{1}{2} = \frac{1}{2}\,\frac{e^{2x} + e^{-2x}}{2} + \frac{1}{2}$$

$$= \frac{e^{2x} + e^{-2x}}{4} + \frac{1}{2} \qquad \square$$

2. (15 points) Find $\int \frac{4y}{y^2+4}\,dy$.

Let $u = y^2+4$

$du = 2y\,dy$

$2du = 4y\,dy$

$$= \int \frac{2}{u}\,du$$

$$= 2\ln|u| + C$$

$$= \boxed{2\ln(y^2+4) + C}$$

OR

Let $y^2+4 = 4\tan^2\theta + 4 = 4\sec^2\theta$

$y^2 = 4\tan^2\theta$

$y = 2\tan\theta$

$dy = 2\sec^2\theta\,d\theta$

$\sqrt{\sec^2\theta} = \sqrt{\frac{1}{4}y^2+1}$

$$= \int \frac{\cancel{8}^{2}\tan\theta}{\cancel{4}\cancel{\sec^2\theta}}\,\cancel{2}\cancel{\sec^2\theta}\,d\theta$$

$$= \int 4\tan\theta\,d\theta$$

$$= 4\ln|\sec\theta| + C = \boxed{4\ln\left(\sqrt{\tfrac{1}{4}y^2+1}\right) + C}$$

3. (15 points) Use integration by parts with cycling to show

$$\int 4\sin(x)\sinh(x)\,dx = 2\sin(x)\cosh(x) - 2\cos(x)\sinh(x) + C.$$

Let $u = 4\sin x$ $\quad v = \cosh x$

$du = 4\cos x\,dx$ $\quad dv = \sinh x\,dx$

$= 4\sin x\cosh x - \int 4\cos x\cosh x\,dx$

Let $u = 4\cos x$ $\quad v = \sinh x$

$du = -4\sin x\,dx$ $\quad dv = \cosh x\,dx$

$= 4\sin x\cosh x - \left(4\cos x\sinh x - \int \sinh x(-4\sin x\,dx)\right)$

$\int 4\sin x\sinh x\,dx = 4\sin x\cosh x - 4\cos x\sinh x - \cancel{\int 4\sin x\sinh x\,dx}$

$+\int 4\sin x\sinh x\,dx$ $\qquad + \cancel{\int 4\sin x\sinh x\,dx}$

$2\int 4\sin x\sinh x\,dx = 4\sin x\cosh x - 4\cos x\sinh x + C$

$\int 4\sin x\sinh x\,dx = 2\sin x\cosh x - 2\cos x\sinh x + C$ $\square$

4. (15 points) Find $\int z^3\sqrt{4-z^2}\,dz$.

Let $4-z^2 = 4-4\sin^2\theta = 4\cos^2\theta$

$z^2 = 4\sin^2\theta$

$z = 2\sin\theta$

$dz = 2\cos\theta\, d\theta$

$\cos^2\theta = 1-\frac{1}{4}z^2$

$\cos\theta = \sqrt{1-\frac{1}{4}z^2}$

$= \int 8\sin^3\theta \sqrt{4\cos^2\theta}\, 2\cos\theta\, d\theta$

$= 32\int \sin^3\theta\cos^2\theta\, d\theta$

$= 32\int (1-\cos^2\theta)\cos^2\theta\sin\theta\, d\theta$

Let $u = \cos\theta$ $\rightarrow$ $u = \sqrt{1-\frac{1}{4}z^2}$

$-du = +\sin\theta\, d\theta$

$= -32\int (1-u^2)u^2\, du$

$= 32\int (u^4-u^2)\,du$

$= 32\left[\frac{1}{5}u^5 - \frac{1}{3}u^3\right] + C$

$= \frac{32}{5}\left(1-\frac{1}{4}z^2\right)^{5/2} - \frac{32}{3}\left(1-\frac{1}{4}z^2\right)^{3/2} + C$

5. (15 points) Find $\int \frac{t^2+2t+2}{t^3+t^2}\,dt$.

$$\frac{t^2+2t+2}{t^2(t+1)} = \frac{A}{t}+\frac{B}{t^2}+\frac{C}{t+1}$$

$$t^2+2t+2 = A(t)(t+1)+B(t+1)+C(t^2)$$

$\boxed{\text{Let } t=0}$

$$2 = \cancel{A(0)(0+1)}+B(1)+\cancel{C(0)}$$

$$2=B$$

$\boxed{\text{Let } t=-1}$

$$1-\cancel{2+2} = \cancel{A(-1)(0)}+\cancel{B(-1+1)}+C\cancel{(1)}$$

$$1=C$$

$$\cancel{t^2+2t+2} = At^2+At+\cancel{2t+2+t^2}$$

$$\cancel{-t^2-2t-2} \qquad \cancel{-2t-2-t^2}$$

$$0=At^2+At$$

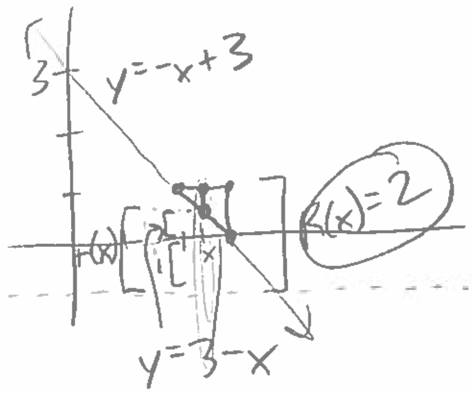
$$A=0$$

$$=\int\left(\frac{2}{t^2}+\frac{1}{t+1}\right)dt$$

$$=\int\left(2t^{-2}+\frac{1}{t+1}\right)dt = -2t^{-1}+\ln|t+1|+C$$

$$=\boxed{\ln|t+1|-\frac{2}{t}+C}$$

6. (15 points) Use the washer method to find an integral which equals the volume of the solid of revolution obtained by rotating the triangle with vertices $(2,1), (3,1), (3,0)$ around the axis $y = -1$.

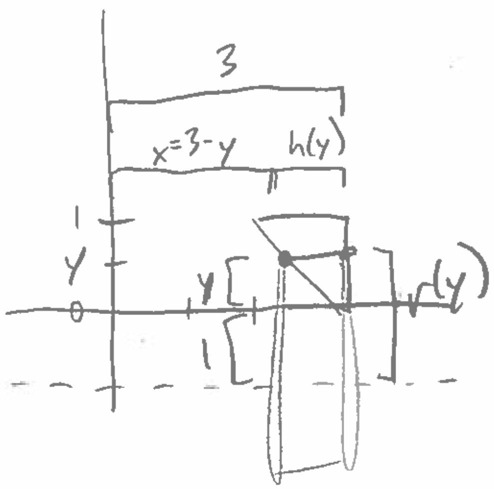


$$\boxed{V = \pi \int_2^3 \left((2)^2 - (4-x)^2\right) dx}$$

$r(x) = 1 + (3-x)$

$r(x) = 4 - x$

7. (15 points) Use the cylindrical shell method to find an integral which equals the volume of the solid of revolution obtained by rotating the triangle with vertices $(2, 1), (3, 1), (3, 0)$ around the axis $y = -1$.



$\cancel{3} - y + h(y) = \cancel{3}$

$h(y) = y$

$r(y) = y + 1$

$$\boxed{V = 2\pi \int_0^1 (y+1)(y)\,dy}$$

8. (15 points) A tank of height 4 meters has a cross-sectional area of $5-y^2$ square meters at a height of y meters. Show that the work required to empty this tank is $10,000\int_0^4(20-4y^2-5y+y^3)\,dy$ joules, assuming that the tank is initially full of salt water weighing 10,000 newtons for every cubic meter. [redacted]

$$A = 5 - y^2$$

$$dV = (5-y^2)\,dy$$

$$dF = 10000\,(5-y^2)\,dy$$

$$dW = 10000\,(5-y^2)(4-y)\,dy$$

$$= 10000\,(20-4y^2-5y+y^3)\,dy$$

$$W = 10000\int_0^4 (20-4y^2-5y+y^3)\,dy \qquad \square$$

9. (BONUS, 5 points) Prove that $\int_0^1 e^x \sinh x \, dx = \frac{1}{4}e^2 - \frac{3}{4}$.

$$\int_0^1 e^x \sinh x \, dx = \int_0^1 e^x \frac{e^x - e^{-x}}{2} \, dx$$

$$= \frac{1}{2}\int_0^1 \left(e^{2x} - 1\right) dx$$

$$= \frac{1}{2}\left[\frac{1}{2}e^{2x} - x\right]_0^1$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}e^2 - 1\right) - \left(\frac{1}{2} - 0\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}e^2 - \frac{3}{2}\right]$$

$$= \frac{1}{4}e^2 - \frac{3}{4} \quad \square$$