① Estimate e with an error no greater than 0.001.

$\frac{1}{1000}$

$$f(x)=e^x: \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots$$

$$|R_n(x)| = \left|\frac{f^{(n+1)}(x_n)}{(n+1)!}(x-0)^{n+1}\right| \quad \text{for } x_n \text{ between } 0 \text{ \& } x.$$

$$|R_n(1)| = \frac{e^{x_n}}{(n+1)!} \cancel{1^{n+1}} \quad \text{for } 0 \le x_n \le 1$$

$$\le \frac{e^1}{(n+1)!}$$

$$< \frac{3}{(n+1)!}$$

$$|R_6(1)| < \frac{3}{7!} = \frac{1}{2\cdot 4\cdot 5\cdot 6\cdot 7} = \frac{1}{1680} < 0.001$$

So

$$e^x \approx 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}$$

$$e^1 \approx 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720}$$

$$\approx 2+0.5+0.1666+0.0416+0.0083+0.0013$$

$$\approx \boxed{2.7178}$$

(Scientific calculator gives 2.7182 ...)

② Estimate $\cos(0.1)$ with an error no greater than 0.0001.

$\frac{1}{10{,}000}$

$$f(x) = \cos(x): \quad \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$

$$|R_n(x)| = \left| \frac{f^{(n+1)}(x_n)}{(n+1)!} (x-0)^{n+1} \right|$$

$\pm \sin(x_n)$ or $\pm \cos(x_n)$

$$\leq \frac{1}{(n+1)!} |x|^{n+1}$$

$$\left|R_n\left(\tfrac{1}{10}\right)\right| \leq \frac{1}{(n+1)!\, 10^{n+1}}$$

$$\left|R_3\left(\tfrac{1}{10}\right)\right| \leq \frac{1}{4!\, 10^4} \leq \frac{1}{10^4} = 0.0001$$

So

$$\cos(x) \approx 1 - \frac{x^2}{2}$$

$$\cos\left(\tfrac{1}{10}\right) \approx 1 - \frac{1}{200}$$

$$\approx 1 - 0.005$$

$$\approx \boxed{0.9950}$$

(Scientific Calculator gives $0.\underbrace{995}0041\ldots$)

③ Estimate $\sin(1)$ with an error no greater than 0.01.

$$f(x) = \sin(x): \quad \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

$$|R_n(x)| = \left| \frac{f^{(n+1)}(x_n)}{(n+1)!} (x-0)^{n+1} \right|$$

$\pm \sin(x_n)$ OR $\pm \cos(x_n)$

$$|R_n(1)| \le \frac{1}{(n+1)!} \cancel{1^{n+1}}$$

$$|R_4(1)| \le \frac{1}{5!} = \frac{1}{120} < \frac{1}{100} = 0.01$$

So

$$\sin x \approx x - \frac{x^3}{6}$$

$$\sin 1 \approx 1 - \frac{1}{6}$$

$$\approx 1 - 0.166$$

$$\approx \boxed{0.833}$$

(Scientific calc gives

0.841...)

(4) Prove that $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$.

Maclaurin Series generated by $\sin(x)$.

$$\lim_{n\to\infty} \left|R_n(x)\right| = \lim_{n\to\infty} \left| \frac{\overbrace{f^{(n+1)}(x_n)}^{\pm\sin(x_n) \text{ or } \pm\cos(x_n)}}{(n+1)!} (x-0)^{n+1} \right|$$

$$\leq \lim_{n\to\infty} \frac{1}{(n+1)!} |x|^{n+1}$$

$$= \lim_{n\to\infty} \frac{|x|^n}{n!}$$

$$= 0$$

Since the limit of the errors is zero,

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}. \quad \square$$

(5) Use the fact that $|\sinh(x_n)| \le |\cosh(x_n)| \le \cosh(x)$ for x_n between 0 and x to prove that $\cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$.

Maclaurin Series generated by $\cosh(x)$.

(Proof:
$f(x) = \cosh(x)$
$f'(x) = \sinh(x)$
$f''(x) = \cosh(x)$
$\to f^{(2k)}(0) = 1$
$\to f^{(2k+1)}(0) = 0$)

$\pm\cosh(x_n)$ OR $\pm\sinh(x_n)$

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left| \frac{f^{(n+1)}(x_n)}{(n+1)!}(x-0)^{n+1} \right|$$

$$= \lim_{n\to\infty} \frac{\cosh(x)}{(n+1)!} |x|^{n+1}$$

$$= \cosh(x) \lim_{n\to\infty} \frac{|x|^n}{n!}$$

$$= \cosh(x)(0)$$

$$= 0$$

Since the limit of the errors is zero,

$$\cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}. \quad \square$$

(6) Reprove $\cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ by using its definition $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ along with the Maclaurin Series for e^x.

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x^k}{k!} + \frac{(-x)^k}{k!} \right)$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \left[\left(\frac{x^{2k}}{(2k)!} + \frac{(-x)^{2k}}{(2k)!} \right) + \left(\frac{x^{2k+1}}{(2k+1)!} + \frac{(-x)^{2k+1}}{(2k+1)!} \right) \right]$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \left[\frac{x^{2k}}{(2k)!} + \frac{x^{2k}}{(2k)!} + \cancel{\frac{x^{2k+1}}{(2k+1)!}} - \cancel{\frac{x^{2k+1}}{(2k+1)!}} \right]$$

$$= \cancel{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\cancel{2} x^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}. \quad \square$$

⑦ Prove that $|\sinh(x_n)| \leq |\cosh(x_n)| \leq \cosh(x)$ for any x_n between 0 and x.

$$|\sinh(x_n)| = \left|\frac{e^{x_n} - e^{-x_n}}{2}\right|$$
$$= \frac{e^{|x_n|} - e^{-|x_n|}}{2}$$
$$\leq \frac{e^{|x_n|} + e^{-|x_n|}}{2}$$
$$= \frac{e^{x_n} + e^{-x_n}}{2}$$
$$= \cosh(x_n) = \cosh|x_n| = |\cosh(x_n)|$$

Since x_n is between 0 and x, $0 \leq |x_n| \leq |x|$.

Since $\frac{d}{dy}[\cosh(y)] = \sinh(y)$ is positive at $y = |x_n|$, it is increasing.

Thus $\cosh|x_n| \leq \cosh|x| = \cosh(x)$. $\square$