(1) Let $f(x) = \frac{1}{1-x}$ with domain $-1<x<1$, and guess a formula for $f^{(k)}(0)$ by computing a few terms. Then show that the Maclaurin Series generated by f converges to f.

$f^{(0)}(x) = (1-x)^{-1} \rightarrow f^{(0)}(0) = 1$

$f^{(1)}(x) = -(1-x)^{-2}(-1)$

$= (1-x)^{-2} \rightarrow f^{(1)}(0) = 1$

$f^{(2)}(x) = +2(1-x)^{-3} \rightarrow f^{(2)}(0) = 2$

$f^{(3)}(x) = +6(1-x)^{-4} \rightarrow f^{(3)}(0) = 6$

$f^{(4)}(x) = +24(1-x)^{-5} \rightarrow f^{(4)}(0) = 24$

$\vdots$

$f^{(k)}(0) = k!$

$$\text{Mac Series} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= \sum_{k=0}^{\infty} \frac{\cancel{k!}}{\cancel{k!}} x^k$$

$$= \sum_{k=0}^{\infty} (1)(x)^k$$

Since $|x|<1$,

$$= \frac{1}{1-x} = f(x). \square$$

② Let $g(x) = \frac{3}{x}$ with domain $0 < x < 6$, and guess a formula for $g^{(k)}(3)$ by computing a few terms. Then show that the Taylor Series generated by g at 3 converges to g.

$$g^{(0)}(x) = 3x^{-1} \rightarrow g^{(0)}(3) = 3\left(\tfrac{1}{3}\right) = 1 = +\frac{0!}{3^0}$$

$$g^{(1)}(x) = 3(-x^{-2}) \rightarrow g^{(1)}(3) = 3\left(-\tfrac{1}{9}\right) = -\frac{1}{3} = -\frac{1!}{3^1}$$

$$g^{(2)}(x) = 3(+2x^{-3}) \rightarrow g^{(2)}(3) = 3(+2)\left(\tfrac{1}{27}\right) = \frac{2}{9} = +\frac{2!}{3^2}$$

$$g^{(3)}(x) = 3(-6x^{-4}) \rightarrow g^{(3)}(3) = 3(-6)\left(\tfrac{1}{81}\right) = \frac{-6}{27} = -\frac{3!}{3^3}$$

$$g^{(4)}(x) = 3(+24x^{-5}) \rightarrow g^{(4)}(3) = 3(+24)\left(\tfrac{1}{243}\right) = +\frac{24}{81} = +\frac{4!}{3^4}$$

$$\vdots$$

$$g^{(k)}(3) = (-1)^k \frac{k!}{3^k}$$

$$\text{Taylor Series at } 3 = \sum_{k=0}^{\infty} \frac{f^{(k)}(3)}{k!}(x-3)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\cancel{k!}/3^k}{\cancel{k!}}(x-3)^k$$

$$= \sum_{k=0}^{\infty} \left(-\tfrac{1}{3}\right)^k (x-3)^k$$

$$= \sum_{k=0}^{\infty} (1)\left(1 - \tfrac{x}{3}\right)^k$$

$$= \frac{1}{\cancel{1-}(\cancel{1}-\frac{x}{3})}$$

$$= \frac{1}{x/3}$$

$$= \frac{3}{x} = g(x). \quad \square$$

Since

$$0 < x < 6$$
$$-3 < x - 3 < 3$$
$$-1 < \tfrac{x}{3} - 1 < 1$$
$$\left|1 - \tfrac{x}{3}\right| = \left|\tfrac{x}{3} - 1\right| < 1 \ldots$$

③ Generate the Maclaurin Series for $\cos x$.

$$f^{(0)}(x) = \cos x \rightarrow f^{(0)}(0) = 1 \rightarrow f^{(2k)}(0) = (-1)^k$$

$$f^{(1)}(x) = -\sin x \rightarrow f^{(1)}(0) = 0$$

$$f^{(2)}(x) = -\cos x \rightarrow f^{(2)}(0) = -1$$

$$f^{(3)}(x) = \sin x \rightarrow f^{(3)}(0) = 0 \rightarrow f^{(2k+1)}(0) = 0$$

$$f^{(4)}(x) = \cos x$$

$$\text{Mac Series} = \sum_{k=0}^{\infty}\left(\frac{f^{(2k)}(0)}{(2k)!}x^{2k} + \cancel{\frac{f^{(2k+1)}(0)}{(2k+1)!}x^{2k+1}}\right)$$

$$= \boxed{\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}} \quad \left(= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$

(4) Generate the Maclaurin Series for $\sinh x$.

$$
\begin{aligned}
f^{(0)}(x) &= \sinh x &\longrightarrow\quad f^{(0)}(0) &= 0 &\rightarrow\quad f^{(2k)}(0) &= 0 \\
f^{(1)}(x) &= \cosh x &\longrightarrow\quad f^{(1)}(0) &= 1 &\rightarrow\quad f^{(2k+1)}(0) &= 1 \\
f^{(2)}(x) &= \sinh x
\end{aligned}
$$

$$\text{Mac Series} = \sum_{k=0}^{\infty}\left(\cancel{\frac{f^{(2k)}(0)}{(2k)!}x^{2k}} + \frac{\cancel{f^{(2k+1)}}(0)}{(2k+1)!}x^{2k+1}\right)$$

$$= \boxed{\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}} \quad \left(= x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\right)$$