9 Does $\sum_{j=2}^{\infty} \frac{e^j}{e^{2j}+1}$ converse or diverge? (Limit Comp)

Similar to $\sum_{j=2}^{\infty} \frac{e^j}{e^{2j}} = \sum_{j=2}^{\infty}\left(\frac{1}{e}\right)^j$ which converges as a geo. series.

$$\lim_{j\to\infty} \frac{\frac{e^j}{e^{2j}+1}}{\frac{1}{e^j}} = \lim_{j\to\infty} \frac{e^{2j}}{e^{2j}+1} = \lim_{j\to\infty} \frac{\cancel{e^{2j}}(1)}{\cancel{e^{2j}}\left(1+\frac{1}{e^{2j}}\right)} = \frac{1}{1+0} = 1$$

Since $\lim_{j\to\infty} \frac{a_n}{b_n} < \infty$, the series $\sum_{j=2}^{\infty} \frac{e^j}{e^{2j}+1}$ $\boxed{\text{converges}}$ as well.

10 Does $\sum_{k=10}^{\infty} \frac{\sin^2 k}{k^3}$ converge or diverge? (Limit Comp)

Compare with $\sum_{k=10}^{\infty} \frac{1}{k^2}$ a convergent p-Series. ($\sum \frac{1}{k^3}$ fails to give a convergent limit)

$$\lim_{k\to\infty} \frac{\frac{\sin^2 k}{k^3}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{\sin^2 k}{k^{\cancel{3}}} \cdot \frac{\cancel{k^2}}{1} = \lim_{k\to\infty} \frac{\sin^2 k}{k} = 0 \text{ (by squeeze thm)}$$

Since $\lim_{j\to\infty} \frac{a_n}{b_n} < \infty$, the series $\sum_{k=10}^{\infty} \frac{\sin^2 k}{k^3}$ also $\boxed{\text{Converges}}$.

(11) Does $\sum_{n=4}^{\infty} \frac{1}{\ln n}$ converse or diverge? (Limit Comp)

Compare with <u>divergent</u> Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

(L'Hopital)

$$\lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x$$ ← diverges to ∞

Since $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, $\sum_{n=4}^{\infty} \frac{1}{\ln n}$ $\boxed{\text{diverges}}$ as well.

(12) Does $\sum_{n=4}^{\infty} \frac{5}{2n+3}$ converse or diverge?

Similar to <u>divergent</u> Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{5}{2n+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{5n}{2n+3} = \frac{5}{2}$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, $\sum_{n=4}^{\infty} \frac{5}{2n+3}$ $\boxed{\text{diverges}}$ as well.

(13) Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2+\cdots+(n-1)+n}$ converge or diverge?

$$\frac{1}{1+2+\cdots+(n-1)+n} = \frac{2}{2\big(1+2+\cdots+(n-1)+n\big)} = \frac{2}{\underset{1}{(1+n)}+\underset{2}{(2+n-1)}+\cdots+\underset{n-1}{(n-1+2)}+\underset{n}{(n+1)}}$$

$$= \frac{2}{\underset{1}{(n+1)}+\underset{2}{(n+1)}+\cdots+\underset{n-1}{(n+1)}+\underset{n}{(n+1)}} = \frac{2}{n(n+1)} \le \frac{2}{n^2}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$ is a convergent p-series, the smaller

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2+\cdots+(n-1)+n}$ also $\boxed{\text{converges}}$.

(14) Does $\sum_{n=0}^{\infty} \frac{2n}{(n^2+1)^2}$ converge or diverge?

OCT

$$\frac{2n}{(n^2+1)^2} \le \frac{2n}{(n^2)^2} = \frac{2n}{n^4} = \frac{2}{n^3}$$

smaller denominator

Since $\sum_{n=1}^{\infty} \frac{2}{n^3}$ is a convergent p-Series, the smaller

$\sum_{n=0}^{\infty} \frac{2n}{(n^2+1)^2}$ is also $\boxed{\text{convergent}}$.

OR

LCT Compare with $\sum_{n=1}^{\infty} \frac{1}{n^3}$ which converges.

$$\lim_{n\to\infty} \frac{\frac{2n}{(n^2+1)^2}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{2n^4}{(n^2+1)^2} = \lim_{n\to\infty} \frac{2n^4}{n^4+2n^2+1} = 2$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, $\sum_{n=0}^{\infty} \frac{2n}{(n^2+1)^2}$ also $\boxed{\text{converges}}$.

15) Does $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+3}}$ converge or diverge?

DCT (Similar to $\sum \sqrt{\frac{n}{n^2}} = \sum \sqrt{\frac{1}{n}} = \sum \frac{1}{n^{1/2}}$ which diverges.)

smaller numerator

$$\sqrt{\frac{n+1}{n^2+3}} \geq \sqrt{\frac{n+1}{n^2+n^2}} = \sqrt{\frac{n+1}{2n^2}} \geq \sqrt{\frac{n+0}{2n^2}} = \sqrt{\frac{1}{2n}} = \frac{1/\sqrt{2}}{n^{1/2}}$$

bigger denominator

Since $\sum_{n=1}^{\infty} \frac{1/\sqrt{2}}{n^{1/2}}$ is a <u>divergent</u> p-Series, the larger $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+3}}$ also $\boxed{\text{diverges}}$.

OR

LCT

Compare with <u>divergent</u> $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$.

$$\lim_{n\to\infty} \frac{\sqrt{\frac{n+1}{n^2+3}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2+3}} = \sqrt{\lim_{n\to\infty} \frac{n^2+n}{n^2+3}} = \sqrt{1} = 1$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, the series $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+3}}$ also $\boxed{\text{diverges}}$.