1. Find the area inside $r = \cos 2\theta$ where $0 \le \theta \le \pi/4$.

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2\, d\theta = \frac{1}{2}\int_0^{\pi/4} \cos^2 2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2}\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{8} + \cancel{\tfrac{1}{8}\sin \pi}\right) - \left(\cancel{0 + \tfrac{1}{8}\sin 0}\right)\right]$$

$$= \boxed{\pi/16}$$

② Find the area bounded by the cardioid $r=1-\cos\theta$.

$$A=\frac{1}{2}\int_{\alpha}^{\beta}(f(\theta))^2\,d\theta=\frac{1}{2}\int_{0}^{2\pi}(1-\cos\theta)^2\,d\theta$$

$$=\frac{1}{2}\int_{0}^{2\pi}\left(1-2\cos\theta+\cos^2\theta\right)d\theta$$

$$=\frac{1}{2}\int_{0}^{2\pi}\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

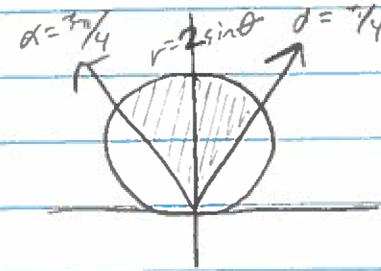
$$=\frac{1}{2}\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_{0}^{2\pi}$$

$$=\frac{1}{2}\left[\left(3\pi-\cancel{2\sin 2\pi}+\cancel{\tfrac{1}{4}\sin 4\pi}\right)-\left(\cancel{0}-\cancel{2\sin 0}+\cancel{\tfrac{1}{4}\sin 0}\right)\right]$$

$$=\boxed{\frac{3\pi}{2}}$$

③ Sketch the region where $|x| \le y \le \sqrt{1-x^2}+1$. Show that its area is $\pi/2 + 1$.

$$y = \sqrt{1-x^2}+1$$
$$y-1 = \sqrt{1-x^2}$$
$$(y-1)^2 = 1-x^2$$
$$x^2+(y-1)^2 = 1$$
$$x^2+y^2-2y+\cancel{1} = \cancel{1}$$
$$r^2 - 2r\sin\theta = 0$$
$$r = 2\sin\theta$$



$$A = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2\,d\theta = \frac{1}{2}\int_{\pi/4}^{3\pi/4} (2\sin\theta)^2\,d\theta$$
$$= \frac{1}{2}\int_{\pi/4}^{3\pi/4} 4\sin^2\theta\,d\theta$$
$$= \int_{\pi/4}^{3\pi/4} 2\sin^2\theta\,d\theta$$
$$= \int_{\pi/4}^{3\pi/4} (1-\cos 2\theta)\,d\theta$$
$$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{3\pi/4}$$
$$= \left(\frac{3\pi}{4} - \frac{1}{2}\sin\frac{3\pi}{2}\right) - \left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right)$$
$$= \frac{3\pi}{4} + \frac{1}{2} - \frac{\pi}{4} + \frac{1}{2}$$
$$= \boxed{\frac{\pi}{2}+1}$$

(4) Find the length of one rotation of the spiral $r = e^{\theta}$.

$f(\theta) = e^{\theta}$

$f'(\theta) = e^{\theta}$

$$L = \int_{\alpha}^{\beta} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2e^{2\theta}}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2}\, e^{\theta}\, d\theta$$

$$= \sqrt{2}\left[e^{\theta}\right]_0^{2\pi}$$

$$= \boxed{\sqrt{2}\left(e^{2\pi} - 1\right)}$$

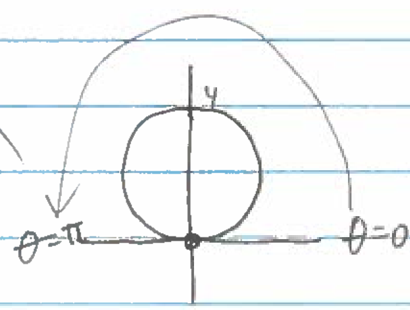
$$= \boxed{\sqrt{2}e^{2\pi} - \sqrt{2}}$$

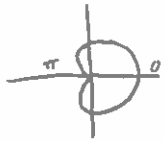
(5) Use the polar arclength formula to show that the circumference of the circle $r=4\sin\theta$ is 4π.

$f(\theta)=4\sin\theta$

$f'(\theta)=4\cos\theta$

$$L=\int_\alpha^\beta \sqrt{(f(\theta))^2+(f'(\theta))^2}\,d\theta$$
$$=\int_0^\pi \sqrt{16\sin^2\theta+16\cos^2\theta}\,d\theta$$
$$=\int_0^\pi \sqrt{16(\cancel{\sin^2\theta+\cos^2\theta})}\,d\theta$$
$$=4\int_0^\pi d\theta$$
$$=4\pi$$





⑥ Show that the length of the cardioid $r = 2 + 2\cos\theta$ is $\int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta = 16$

$f(\theta) = 2 + 2\cos\theta$

$f'(\theta) = -2\sin\theta$

$$L = \int_0^{2\pi} \sqrt{(2+2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{16\cos^2(\theta/2)}\, d\theta$$

$$= \int_0^{2\pi} 4|\cos(\theta/2)|\, d\theta$$

$$= \int_{-\pi}^{\pi} 4|\cos(\theta/2)|\, d\theta$$

$$= \int_{-\pi}^{\pi} 4\cos(\theta/2)\, d\theta$$

(Cosine is positive from $-\pi/2$ to $\pi/2$)

$$= \left[8\sin(\theta/2)\right]_{-\pi}^{\pi}$$

$$= (8) - (-8)$$

$$= \boxed{16}$$

⑦ Prove that if $x=f(\theta)\cos\theta$ and $y=f(\theta)\sin\theta$ then
$$\int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta = \int_\alpha^\beta \sqrt{(f(\theta))^2+(f'(\theta))^2}\,d\theta.$$

$$x=f(\theta)\cos\theta$$
$$dx/d\theta=(\cos\theta)(f'(\theta))+(f(\theta))(-\sin\theta)$$
$$(dx/d\theta)^2=(f'(\theta))^2\cos^2\theta - 2f(\theta)f'(\theta)\sin\theta\cos\theta+(f(\theta))^2\sin^2\theta$$

$$y=f(\theta)\sin\theta$$
$$dy/d\theta=(\sin\theta)(f'(\theta))+(f(\theta))(\cos\theta)$$
$$(dy/d\theta)^2=(f'(\theta))^2\sin^2\theta+2f(\theta)f'(\theta)\sin\theta\cos\theta+(f(\theta))^2\cos^2\theta$$

$$(dx/d\theta)^2+(dy/d\theta)^2=(f'(\theta))^2\cancel{(\cos^2\theta+\sin^2\theta)}+(f(\theta))^2\cancel{(\sin^2\theta+\cos^2\theta)}$$
$$=(f(\theta))^2+(f'(\theta))^2 \quad \square$$