⑥ Find $\int \frac{\sqrt{x^2-16}}{x}\,dx$ where $x \geq 4$.

Let $x^2-16 = 16\sec^2\theta - 16 = 16\tan^2\theta \rightarrow \tan^2\theta = \frac{1}{16}x^2 - 1$

$x^2 = 16\sec^2\theta$

$x = 4\sec\theta \longrightarrow \frac{x}{4} = \sec\theta$

$dx = 4\sec\theta\tan\theta\,d\theta \qquad \theta = \sec^{\leftarrow}\left(\frac{x}{4}\right)$

$= \int \frac{\sqrt{16\tan^2\theta}}{\cancel{4\sec\theta}} \cancel{4\sec\theta}\tan\theta\,d\theta$

$= \int 4\tan^2\theta\,d\theta$

$= \int 4\sec^2\theta - 4\,d\theta$

$= 4\tan\theta - 4\theta + C$

$= \boxed{4\sqrt{\frac{1}{16}x^2-1} - 4\sec^{\leftarrow}\left(\frac{x}{4}\right) + C}$

OR

$= \boxed{\sqrt{x^2-16} - 4\sec^{\leftarrow}\left(\frac{x}{4}\right) + C}$

7 Find $\int \frac{1}{\sqrt{4t^2-1}}\,dt$ where $t > \frac{1}{2}$.

Let $4t^2 - 1 = \sec^2\theta - 1 = \tan^2\theta \rightarrow \tan\theta = \sqrt{4t^2-1}$

$4t^2 = \sec^2\theta$

$2t = \sec\theta$

$t = \frac{1}{2}\sec\theta$

$dt = \frac{1}{2}\sec\theta\tan\theta\, d\theta$

$$= \int \frac{1}{\cancel{\sqrt{\tan^2\theta}}}\,\frac{1}{2}\sec\theta\,\cancel{\tan\theta}\,d\theta$$

$$= \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\frac{1}{2}\ln\left(2t + \sqrt{4t^2-1}\right) + C}$$

8. Find $\int \frac{2}{\sqrt{1-4x^2}}\,dx$ without a trig sub.

(Looks like $\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{\leftarrow}(u) + C$.)

Let $4x^2 = u^2$

$2x = u$

$2dx = du$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{\leftarrow}(u) + C$$

$$= \boxed{\sin^{\leftarrow}(2x) + C}$$

(9) Find $\int \frac{2}{4+9x^2}\,dx$ without a trig sub.

$\left(\text{Looks like } \int \frac{1}{1+u^2}\,du = \tan^{\leftarrow}(u) + C. \right)$

$$= \int \frac{\cancel{2}^{1}}{\cancel{4}_{2}} \frac{1}{1+\frac{9}{4}x^2}\,dx$$

Let $\frac{9}{4}x^2 = u^2$

$\frac{3}{2}x = u$

$x = \frac{2}{3}u$

$dx = \frac{2}{3}\,du$

$$= \frac{1}{\cancel{2}} \frac{\cancel{2}}{3} \int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{3}\tan^{\leftarrow}(u) + C$$

$$= \boxed{\frac{1}{3}\tan^{\leftarrow}\left(\frac{3}{2}x\right) + C}$$

(10) Find $\int \frac{1}{\sqrt{9+y^2}}\, dy$.

Let $9+y^2 = 9+9\tan^2\theta = 9\sec^2\theta$ $\quad \sec\theta = \sqrt{1+\frac{1}{9}y^2}$

$y^2 = 9\tan^2\theta$

$y = 3\tan\theta \quad \tan\theta = \frac{y}{3}$

$dy = 3\sec^2\theta\, d\theta$

$$= \int \frac{1}{\cancel{\sqrt{9\sec^2\theta}}} \cancel{3}\sec^{\cancel{2}}\theta\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\ln\left|\sqrt{1+\frac{1}{9}y^2} + \frac{y}{3}\right| + C}$$

$$= \boxed{\ln\left(\sqrt{1+\frac{1}{9}y^2} + \frac{1}{3}y\right) + C}$$

(11) Find $\int \frac{1}{x\sqrt{4x^2-1}}\,dx$ where $x > \frac{1}{2}$.

Let $4x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$

$4x^2 = \sec^2\theta$

$2x = \sec\theta \rightarrow \theta = \sec^{\leftarrow}(2x)$

$x = \frac{1}{2}\sec\theta$

$dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$

$$= \int \frac{1}{\cancel{\frac{1}{2}}\cancel{\sec\theta}\sqrt{\cancel{\tan^2\theta}}}\cancel{\tfrac{1}{2}}\cancel{\sec\theta}\cancel{\tan\theta}\,d\theta$$

$$= \theta + C$$

$$= \boxed{\sec^{\leftarrow}(2x) + C}$$

OR

Let $4x^2 - 1 = u^2 - 1$

$4x^2 = u^2$

$2x = u$

$x = u/2$

$dx = \frac{1}{2}\,du$

$$= \int \frac{1}{\cancel{u/2}\sqrt{u^2-1}}\cancel{\tfrac{1}{2}}\,du = \sec^{\leftarrow}(u) + C = \boxed{\sec^{\leftarrow}(2x) + C}$$