① Find $\int \sin^4 x \cos^3 x \, dx$

$$= \int \sin^4 x \cos^2 x \cos x \, dx$$

$$= \int \sin^4 x (1 - \sin^2 x) \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

$$= \int u^4 (1 - u^2) \, du$$

$$= \int u^4 - u^6 \, du$$

$$= \frac{1}{5} u^5 - \frac{1}{7} u^7 + C$$

$$= \boxed{\frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C}$$

(2) Find $\int \sin^5\theta \cos^2\theta \, d\theta$.

$$= \int \sin^4\theta \cos^2\theta \sin\theta \, d\theta$$

$$= \int (1-\cos^2\theta)^2 \cos^2\theta \sin\theta \, d\theta$$

Let $u = \cos\theta$

$du = -\sin\theta \, d\theta$

$-du = +\sin\theta \, d\theta$

$$= \int (1-u^2)^2 u^2 (-du)$$

$$= \int (1-2u^2+u^4) u^2 (-du)$$

$$= \int -u^2 + 2u^4 - u^6 \, du$$

$$= -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C$$

$$= \boxed{-\frac{1}{3}\cos^3\theta + \frac{2}{5}\cos^5\theta - \frac{1}{7}\cos^7\theta + C}$$

(3) Find $\int \sin^2 x \, dx$.

$$= \int \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$$

$$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin 2x + C}$$

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(4) Find $\int \cos^4 y \, dy$.

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos 2y\right)^2 dy$$

$$= \int \frac{1}{4} + \frac{1}{2}\cos 2y + \frac{1}{4}\cos^2 2y \, dy$$

$$= \int \frac{1}{4} + \frac{1}{2}\cos 2y + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4y\right) dy$$

$$= \int \frac{3}{8} + \frac{1}{2}\cos 2y + \frac{1}{8}\cos 4y \, dy$$

$$= \boxed{\frac{3}{8}y + \frac{1}{4}\sin 2y + \frac{1}{32}\sin 4y + C}$$

(5) Find $\int \tan^2 t \sec^4 t \, dt$.

$$= \int \tan^2 t \sec^2 t \sec^2 t \, dt$$

$$= \int \tan^2 t \left(\tan^2 t + 1\right) \sec^2 t \, dt$$

Let $u = \tan t$

$du = \sec^2 t \, dt$

$$= \int u^2 (u^2 + 1) \, du$$

$$= \int u^4 + u^2 \, du$$

$$= \frac{1}{5} u^5 + \frac{1}{3} u^3 + C$$

$$= \boxed{\frac{1}{5} \tan^5 t + \frac{1}{3} \tan^3 t + C}$$

(6) Prove $\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x+\tan x| + C$.

$$\int \sec^3 x\,dx = \int \sec x\,\sec^2 x\,dx$$

Let $u = \sec x$ $\quad v = \tan x$

$du = \sec x\tan x\,dx$ $\quad dv = \sec^2 x\,dx$

$$= \sec x\tan x - \int \sec x\tan^2 x\,dx$$

$$= \sec x\tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$\int \sec^3 x\,dx = \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$+\int \sec^3 x\,dx$ $\quad +\int \sec^3 x\,dx$

$$2\int \sec^3 x\,dx = \sec x\tan x + \ln|\sec x+\tan x| + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x+\tan x| + C. \quad \square$$