6 Find $\int \sin(2x)\cos(4x)\,dx$.

Easy Way

Let $u=\cos(4x)$ $\quad v=-\frac{1}{2}\cos(2x)$

$du=-4\sin(4x)\,dx$ $\quad dv=\sin(2x)\,dx$

$$= -\frac{1}{2}\cos(4x)\cos(2x) - \int\left(-\frac{1}{2}\cos(2x)\right)\left(-4\sin(4x)\right)dx$$

$$= -\frac{1}{2}\cos(4x)\cos(2x) - \int 2\cos(2x)\sin(4x)\,dx$$

Let $u=\sin(4x)$ $\quad v=\sin(2x)$

$du=4\cos(4x)\,dx$ $\quad dv=2\cos(2x)\,dx$

$$= -\frac{1}{2}\cos(4x)\cos(2x) - \left[\sin(4x)\sin(2x) - \int 4\sin(2x)\cos(4x)\,dx\right]$$

$$\int \sin(2x)\cos(4x)\,dx = -\frac{1}{2}\cos(4x)\cos(2x) - \sin(4x)\sin(2x) + \cancel{4\int \sin(2x)\cos(4x)\,dx}$$

$-4\int \sim dx$ $\qquad -4\int \sim dx$

$$-3\int \sim dx = -\frac{1}{2}\cos(4x)\cos(2x) - \sin(4x)\sin(2x) + C$$

$$\int \sin(2x)\cos(4x)\,dx = \boxed{\frac{1}{6}\cos(4x)\cos(2x) + \frac{1}{3}\sin(4x)\sin(2x) + C}$$

Hard way

Let $u = \sin(2x)$ $\quad v = \frac{1}{4}\sin(4x)$

$du = 2\cos(2x)\,dx$ $\quad dv = \cos(4x)\,dx$

$$= \sin(2x)\frac{1}{4}\sin(4x) - \int \frac{1}{4}\sin(4x)\, 2\cos(2x)\,dx$$

$$= \frac{1}{4}\sin(2x)\sin(4x) - \int \frac{1}{2}\cos(2x)\sin(4x)\,dx$$

Let $u = \frac{1}{2}\cos(2x)$ $\quad v = -\frac{1}{4}\cos(4x)$

$du = -\sin(2x)\,dx$ $\quad dv = \sin(4x)\,dx$

$$= \frac{1}{4}\sin(2x)\sin(4x) - \left[\frac{1}{2}\cos(2x)\left(-\frac{1}{4}\right)\cos(4x) - \int -\frac{1}{4}\cos(4x)(-\sin(2x))\,dx\right]$$

$$\int \sin(2x)\cos(4x)\,dx = \frac{1}{4}\sin(2x)\sin(4x) + \frac{1}{8}\cos(2x)\cos(4x) + \cancel{\frac{1}{4}\int \sin(2x)\cos(4x)\,dx}$$

$$-\frac{1}{4}\int \text{——}\,dx$$

$$-\frac{1}{4}\int \text{——}\,dx$$

$$\frac{3}{4}\int \sin(2x)\cos(4x)\,dx = \frac{1}{4}\sin(2x)\sin(4x) + \frac{1}{8}\cos(2x)\cos(4x) + C$$

$$\int \sin(2x)\cos(4x)\,dx = \frac{4}{3}\left(\frac{1}{4}\sin(2x)\sin(4x) + \frac{1}{8}\cos(2x)\cos(4x)\right) + C$$

$$= \boxed{\frac{1}{3}\sin(2x)\sin(4x) + \frac{1}{6}\cos(2x)\cos(4x) + C}$$

7 Compute $\int_1^e x \ln x \, dx$.

Let $u = \ln x \quad v = \frac{1}{2}x^2$

$du = \frac{1}{x}dx \quad dv = x\, dx$

$$\int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^{\cancel{2}} \cancel{\frac{1}{x}} \, dx$$

$$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2} x \, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

$$\int_1^e x \ln x \, dx = \left[\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right]_1^e$$

$$= \left[\frac{1}{2}e^2 \cancel{\ln} e - \frac{1}{4}e^2\right] - \left[\cancel{\frac{1}{2}1^2 \ln 1} - \frac{1}{4}\cancel{1^2}\right]$$

$$= \boxed{\frac{1}{4}e^2 - \frac{1}{4}}$$

8 Find $\int x^4 e^x dx$.

$u = x^4 \quad v = e^x$

$du = 4x^3 dx \quad dv = e^x dx$

$$= x^4 e^x - \int 4x^3 e^x dx$$

$u = 4x^3 \quad v = e^x$

$du = 12x^2 \quad dv = e^x dx$

$$= x^4 e^x - 4x^3 e^x + \int 12x^2 e^x dx$$

$u = 12x^2 \quad v = e^x$

$du = 24x dx \quad dv = e^x dx$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - \int 24x e^x dx$$

$u = 24x \quad v = e^x$

$du = 24 dx \quad dv = e^x dx$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + \int 24 e^x dx$$

$$= \boxed{x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24 e^x + C}$$

(9) Prove $\int \cos^{n+2} x \, dx = \dfrac{\cos^{n+1} x \sin x}{n+2} + \dfrac{n+1}{n+2} \int \cos^n x \, dx.$

$$\frac{d}{dx}\left[\frac{\cos^{n+1} x \sin x}{n+2} + \frac{n+1}{n+2} \int \cos^n x \, dx\right]$$

$$= \frac{1}{n+2}\left(\sin x\left((n+1)\cos^n x(-\sin x)\right) + \cos^{n+1} x(\cos x)\right) + \frac{n+1}{n+2}\cos^n x$$

$$= \frac{1}{n+2}\cos^{n+2} x - \frac{n+1}{n+2}\cos^n x \sin^2 x + \frac{n+1}{n+2}\cos^n x$$

$$= \frac{1}{n+2}\cos^{n+2} x + \frac{n+1}{n+2}\cos^n x\left(-\sin^2 x + 1\right)$$

$$= \frac{1}{n+2}\cos^{n+2} x + \frac{n+1}{n+2}\cos^n x\left(\cos^2 x\right)$$

$$= \frac{1}{n+2}\cos^{n+2} x + \frac{n+1}{n+2}\cos^{n+2} x$$

$$= \cos^{n+2} x.$$

Thus $\dfrac{\cos^{n+1} x \sin x}{n+2} + \dfrac{n+1}{n+2}\int \cos^n x \, dx$ is the general antiderivative of $\cos^{n+2} x$. $\square$

(10) Find $\int \cos^4 x\, dx$ using #9 formula.

$$= \int \cos^{2+2} x\, dx$$

$$= \frac{\cos^{2+1} x \sin x}{2+2} + \frac{2+1}{2+2} \int \cos^2 x\, dx$$

$$= \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \int \cos^{0+2} x\, dx$$

$$= \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \left[\frac{\cos^{0+1} x \sin x}{0+2} + \frac{0+1}{0+2} \int \cancel{\cos^0} x\, dx \right]$$

$$= \boxed{\frac{\cos^3 x \sin x}{4} + \frac{3 \cos x \sin x}{8} + \frac{3}{8} x + 1}$$

11 Find $\int x \cosh x \, dx$.

Let $u = x$ $\quad v = \sinh x$

$du = dx \quad dv = \cosh x \, dx$

$= x \sinh x - \int \sinh x \, dx$

$= \boxed{x \sinh x - \cosh x + C}$

(12) Find $\int e^{\theta} \sin\theta \, d\theta$.

Let $u = \sin\theta$ $\quad v = e^{\theta}$

$du = \cos\theta \, d\theta$ $\quad dv = e^{\theta} d\theta$

$$= e^{\theta}\sin\theta - \int e^{\theta}\cos\theta \, d\theta$$

Let $u = \cos\theta$ $\quad v = e^{\theta}$

$du = -\sin\theta$ $\quad dv = e^{\theta} d\theta$

$$= e^{\theta}\sin\theta - \left[e^{\theta}\cos\theta - \int e^{\theta}(-\sin\theta)\,d\theta\right]$$

$$\int e^{\theta}\sin\theta \, d\theta = e^{\theta}\sin\theta - e^{\theta}\cos\theta - \int e^{\theta}\sin\theta \, d\theta$$

$$2\int e^{\theta}\sin\theta \, d\theta = e^{\theta}\sin\theta - e^{\theta}\cos\theta + C$$

$$\int e^{\theta}\sin\theta \, d\theta = \boxed{\frac{e^{\theta}\sin\theta - e^{\theta}\cos\theta}{2} + C}$$

(Can also start by using

$u = e^{\theta}$ $\quad v = -\cos\theta$

$du = e^{\theta} d\theta$ $\quad dv = \sin\theta \, d\theta$

instead.)